Some simplifications of Coulomb's active earth pressure theory

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Coulomb's earth pressure theory was published in 1776. Since that time this theory has been extended and embellished by many researchers. To this day Coulomb's theory still holds good for the calculation of active earth pressure for idealised smooth walls and rough walls where the angle of wall friction is positive.

One criticism of Coulomb's theory, especially when applied to rough walls or sloping backfill, is that the resulting equations are arithmetically complex and tedious to evaluate. This note reconsiders Coulomb's theory and suggests simplifications which yield more manageable equations which can be used without significant loss of accuracy.

Introduction

IN HIS SOLUTION of the active earth pressure problem, Coulomb considered the equilibrium of a wedge of soil bounded by a smooth vertical wall, Fig. 1. The soil wedge was considered to undergo rigid body motion along a planar rupture surface inclined at θ to the horizontal. Coulomb tacitly assumed that the strain along the planar rupture surface was sufficient to fully mobilise the shear strength of the soil, in the case of a cohesionless soil it was assumed that ϕ, the angle of shearing resistance, was fully mobilised.

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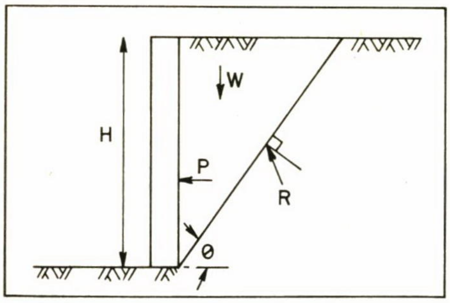

Fig. 1. The Coulomb soil wedge

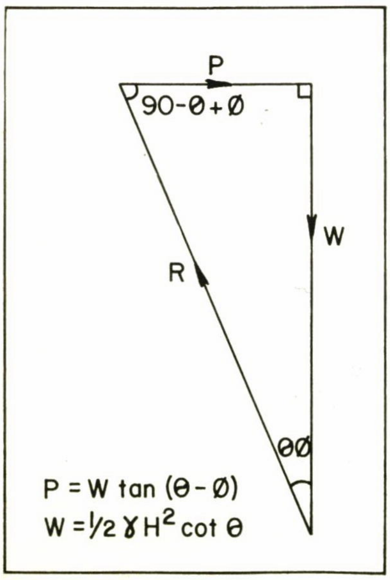

Fig. 2. The triangle of forces

From the triangle of forces for the soil wedge, Fig. 2, an expression may be derived for the total earth thrust P:

$$P = \tfrac{1}{2} K\gamma H^2 \qquad \dots (1a)$$

where $$K = \cot\theta \tan(\theta - \phi) \qquad \dots (1b)$$

By differentiating Eqn. *1a* with respect to θ it is found that for a maximum value of P:

$$\theta = 45^\circ + \phi/2 \qquad \dots (2a)$$

$$K_a = \frac{1 - \sin\phi}{1 + \sin\phi} \qquad \dots (2b)$$

where K_a is the coefficient of active earth pressure.

In the original Coulomb analysis the inclination of the rupture plane was defined by the horizontal distance from the top of the wall to the top of the emergent plane of rupture with the shear strength of the soil being represented by a coefficient of friction. It was not until 1794 that Woltmann defined the shear strength of the soil in terms of tan ϕ. In 1802 Prony simplified Coulomb's work and presented it in the now familar trigonometric form.

The effects of wall friction were considered qualitatively by Coulomb, but it was not until 1808 that Mayniel, and subsequently Francois in 1820, made a quantitative assessment of δ, the angle of wall friction, which is expressed in Eqn. 5. In 1906 Müller-Breslau extended Coulomb's work to a completely general solution, for cohesionless soil, which allowed for wall friction, sloping backfill, and walls with inclined backs.

The Coulomb-Mayniel solution

In 1808 Mayniel extended the work of Coulomb to take into account the effects of wall friction. The expression for the general earth pressure coefficient, again derived from consideration of the equilibrium of a wedge of soil is given by the expression:

$$K = \cot\theta \sin(\theta - \phi) \sec(\delta + \phi - \theta) \qquad \dots (3)$$

By differentiating with respect to θ to maximise K, the value of θ is:

$$\theta = \tan^{-1}\left[\tan\phi + \sec\phi\sqrt{\frac{\tan\phi}{\tan\phi + \tan\delta}}\right] \dots (4)$$

Substituting this value of θ into Eqn. *3* gives the familiar expression:

$$K_a = \left[\frac{\cos\phi}{\sqrt{\cos\delta} + \sqrt{\sin(\delta + \phi)\sin\phi}}\right]^2 \qquad \dots (5)$$

It has been shown (Krey, 1936; Caquot & Kerisel, 1956) that for active earth pressure no great inaccuracy is involved by assuming a planar rupture surface. However, work by many researchers (Kötter, 1903; Prandtl, 1920 & 1927; Sokolovskii, 1942) has indicated that the inclination of the rupture plane given by Eqn. *4* is only valid in the vicinity of the heel of the retaining wall. The "true" rupture surface is in fact curvilinear (Fig. 3), with the emergent rupture plane being inclined at $45^\circ + \phi/2$ to the horizontal. It is found that by taking a psuedo planar rupture plane inclined at $45^\circ + \phi/2$ to the horizontal, as opposed to θ as defined in Eqn.

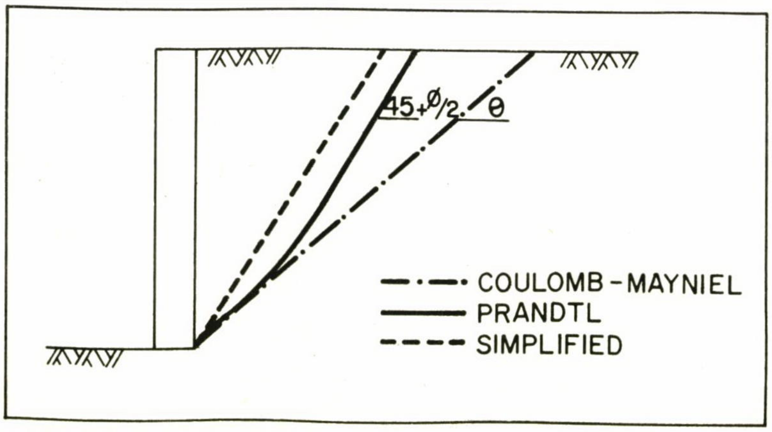

Fig. 3. Wedge rupture surfaces

4 that Eqn. 5 reduces to:

$$K_a = \frac{1 - \sin\phi}{\cos\delta + \sin(\delta + \phi)} \quad \dots (6)$$

It can be seen that substitution of $\delta = 0$ into Eqn. 6, leads immediately to the value of K_a given in Eqn. 2b. For values of $0 < \delta < 2/3\,\phi$, the additional error induced by use of Eqn. 6 is less than 1.5%. The variation of error with the ratio of δ/ϕ is shown in Fig. 4.

The Coulomb-Müller-Breslau solution

In 1906 Müller-Breslau published a general solution for active thrust in a cohesionless soil. For a rough wall with a vertical back and backfill sloping at an angle β the coefficient of active earth pressure is given by the expression:

$$K_a = \frac{\cos^2\phi}{\cos\delta\left[1 + \sqrt{\dfrac{\sin(\phi-\beta)\sin(\delta+\phi)}{\cos\beta\cos\delta}}\right]^2} \quad \dots (7)$$

It was found that substituting $\theta = 45 + \phi/2$ in the generic expression for Eqn. 7 did not lead to any worthwhile simplification.

For the special case when $\delta = 0$, i.e. a smooth wall, Eqn. 7 reduces to Eqn. 8. It is to be noted that the idealised "smooth wall" assumption is of practical application in the design of walls and sheet piles subject to vibration, or coated with a bituminous material that would act as a slip layer. The smooth wall analysis also has application in the design of anchored sheet pile walls where the vertical component of the tie force can lead to downward vertical movement of the pile, (Broms & Stille, 1976), thus negating the effects of positive wall friction.

$$K_a = \frac{\cos^2\phi}{\left[1 + \sqrt{\dfrac{\sin(\phi-\beta)\sin\phi}{\cos\beta}}\right]^2} \quad \dots (8)$$

Eqn. 8 is derived from the maximised value of the generic equation, Eq. 9, by substitution of the value of θ from Eqn. 10.

$$K = \cos\beta\cos\theta\tan(\theta-\phi)\ \mathrm{cosec}\,(\theta-\beta) \quad \dots (9)$$

$$\theta = \tan^{-1}\left[\tan\phi + \sec\phi\sqrt{\frac{\tan\phi - \tan\beta}{\tan\phi}}\right] \quad \dots (10)$$

By substituting the value $\theta = 45° + \phi/2$ into Eqn. 9, the expression for K_a is much simplified as shown in Eqn. 11.

$$K_a = \frac{1 - \sin\phi}{1 + \sin\phi - \cos\phi\tan\beta} \quad \dots (11)$$

From inspection, it can be seen that putting $\beta = 0$ in Eqn. 11 results in the value of K_a given by Eqn. 2b.

The error induced by use of Eqn. 11 for $|\beta/\phi| < 2/3\,\phi$ is less than 1.5% for negative values of β and less than 4.5% for positive values of β. The variation of error

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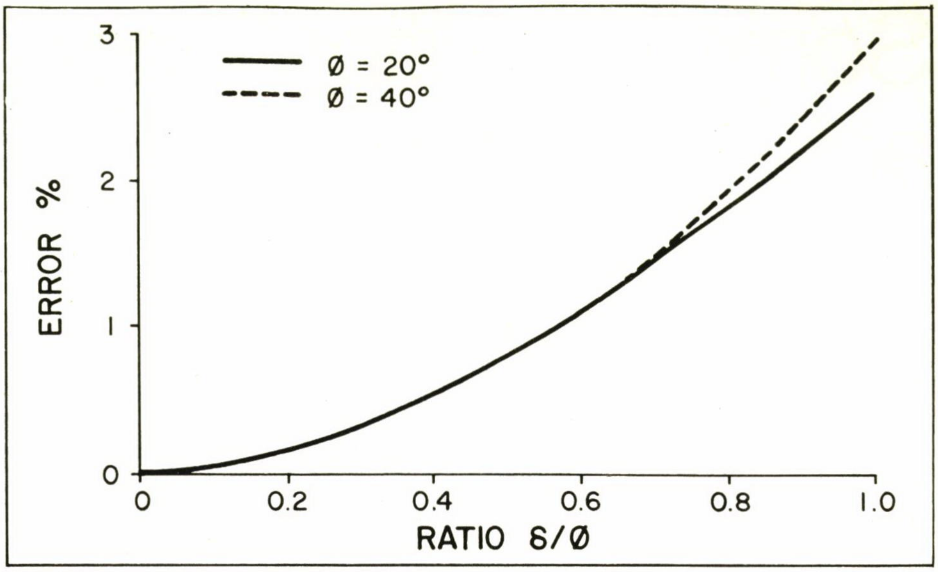

Fig. 4. Variation of error with δ/ϕ ratio

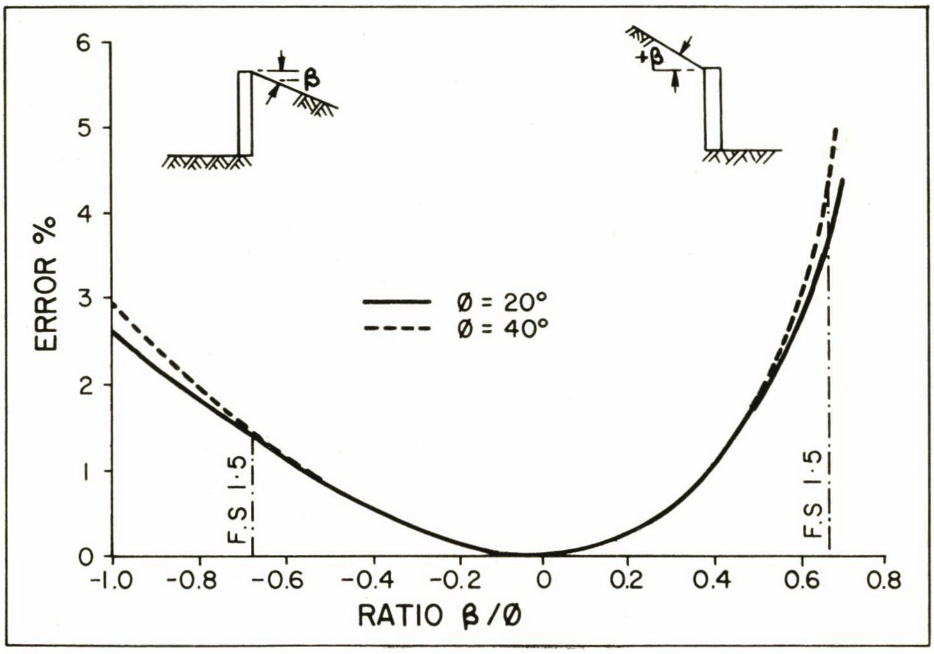

Fig. 5. Variation of error with β/ϕ ratio

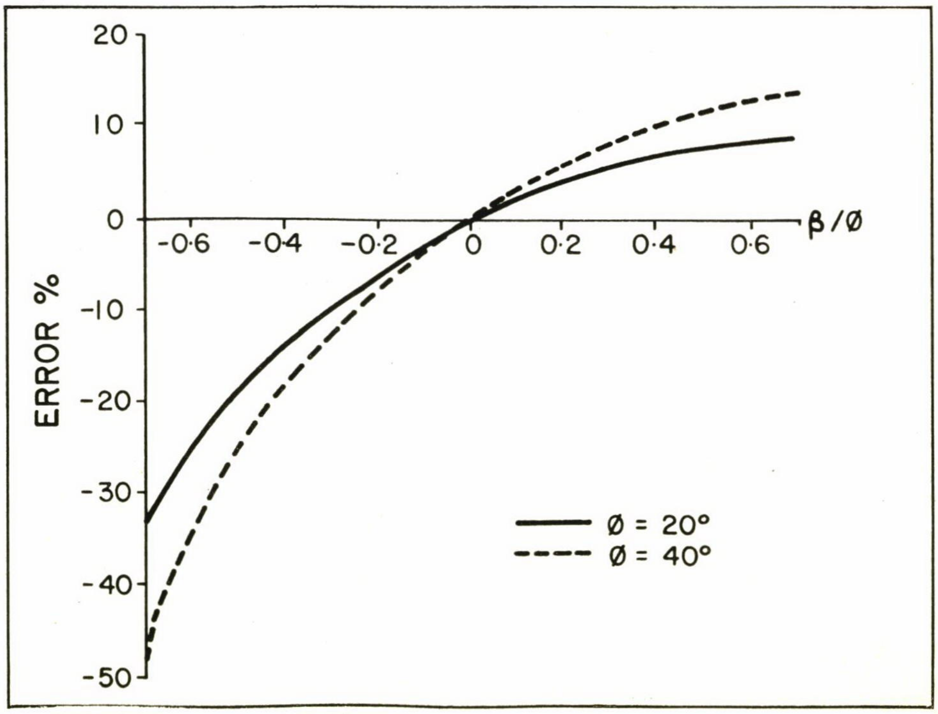

Fig. 6. Variation of error with β/ϕ ratio (Rankine theory)

friction of the piled block—i.e. a deduction of 180MN derived from consideration of the piles acting as a block, the area acting in friction being the length of the pile shafts (13m) multiplied by the building perimeter. In arriving at this value an average undrained shear strength of 177kN/m^2 and an adhesion factor of 0.3 have been used.

The results of the analyses are shown in Fig. 4 where the computed settlement has been plotted with respect to the ratio y/H. It is immediately apparent that if no deduction is made for the skin friction component the settlement is consistently over-estimated. However, when the load carried in skin friction is allowed for, agreement between computed and measured settlements is obtained for $y/H=0.7$ (transversely isotropic model) and $y/H=0.77$ (isotropic model).

Furthermore, the computed settlement is relatively insensitive over the range of $y/H=0.5$ (over-estimate of 20%) to $y/H=1.0$ (underestimate of 20%). In Fig. 5 computed ($y/H=0.7$) and measured settlement profiles show satisfactory agreement; and the effect of ignoring the stiffness of the structure is clearly demonstrated.

Comparative study

In order to test the validity of the results obtained above a similar approach has been adopted in re-analysing results presented for a 14-storey residential block on London Clay by Hooper and Wood[3]. The building is approximately rectangular 30m long by 18m wide with the lower two storeys of cast in-situ reinforced concrete and the remaining superstructure of precast units with in-situ stitches. The foundation comprises 48 under-reamed concrete piles varying in length from 17.8m to 19.4m with shaft diameters of 0.76m and 0.91m, and base diameters from 1.2m to 2.0m.

In this case the presence of numerous shear walls in the superstructure forming an "egg-box" type of arrangement gave rise to a 4.0m thick raft of equivalent bending stiffness. Using the same technique, i.e. considering the skin friction to act over an area given by the building perimeter and the average shaft length of the piles into clay, an adhesion factor of 0.3 and an average undrained shear strength of 140kN/m^2, a deduction of 48MN has been made to the total gross load of 106MN (considered to be uniformly distributed over the plan area of the building).

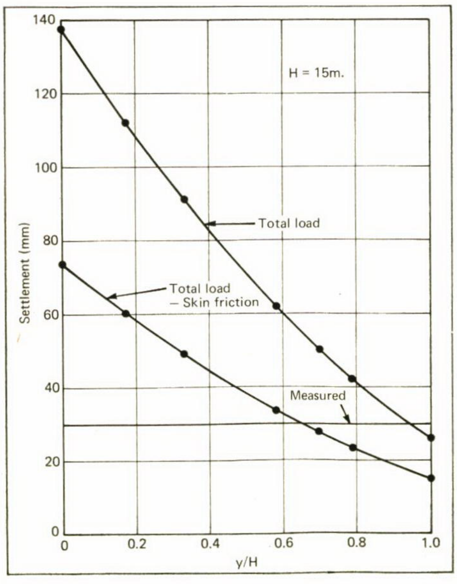

Fig. 6. Computed settlements comparative study

The same soil parameters as those adopted by Hooper and Wood[3] for the transversely isotropic model have been used, namely:

Gravel 4m thick $E'_v=100.0\text{MN/m}^2$

London Clay 17m thick $E'_v=6.0+1.0z\text{MN/m}^2$

Woolwich and Reading Beds 14m thick $E'_v=200.0\text{MN/m}^2$,

and for all layers $E'_h = 2.3\,E'_v$,

$G'_{vh} = 0.66\,E'_v$,

$\nu'_{vh} = 0.1$ and $\nu'_{hh} = -0.15$

Again, the vertical displacement of the underlying Thanet Sand and Chalk strata has been assumed negligible and ignored.

The results are shown in Fig. 6, where it is apparent that if no deduction for skin friction is made then measured (30mm) and computed settlements agree at a value of $y/H = 0.94$. However, when the curve obtained from the modified load is considered agreement occurs for $y/H = 0.65$; but the computed settlement is relatively insensitive over the range of $y/H = 0.54$ to 0.8.

Conclusions

One rather obvious general conclusion that may be drawn from the analyses is that when comparing measured and computed settlements it is essential to take account of the stiffness of the structure. Indeed if this is not done two different systems are being compared.

It may also be tentatively concluded from this study of two buildings of dissimilar bending stiffnesses that, for piled buildings on London Clay, an equivalent raft positioned at a depth into clay of between 0.5 and 0.8 of the pile penetration into clay should yield a good estimate of the settlement performance of the structure (both in terms of total and differential movement), provided that the applied load is reduced to take into account the component of ground reaction acting in skin friction.

Acknowledgements

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References

1. *Burland, J. B., Butler, F. G. & Dunican, P. (1966):* "The behaviour and design of large diameter bored piles in stiff clays". Proc. Symp. Large Bored Piles, London. Inst. Civ. Eng. 51-71.
2. *Whitaker, T. (1976):* The design of piled foundations. 2nd Ed., Pergamon Press.
3. *Hooper, J. A. & Wood, L A. (1977):* "Comparative behaviour of raft and piled foundations". To appear in Proc. 9th Int. Conf. Soil Mech. Fdn. Engng , Tokyo.
4. *Neasor, E. O. & Williams, G. M J. (1962):* "Features in the design and construction of the Shell Centre, London". Proc. Inst. Civ. Eng., Vol. **21** 475-502.
5. *Norton, K. & Au, E. (1974):* "Settlement observation on eight structures in London". Proc. Conf. Settlement of Structures Cambridge, Pentech Press, 183-203.
6. *Wood, L. A & Larnach, W. J. (1974):* "The effects of soil-structure interaction on raft foundations". Proc. Conf. Settlement of Structures, Cambridge, Pentech Press, 460-470.
7. *Wood, L. A. & Larnach, W. J.* (1975): The interactive behaviour of a soil-structure system and its effect on settlements". Proc, Symp. Geot. Struct, Univ. N.S.W.
8. *Wood, L. A.* (1977): "The economic analysis of raft foundations". Int. J. Numerical and Analytical Methods in Geomechanics, Wiley, Vol. **1**, No. 4, pp. 397-405.

Some simplifications of Coulomb's theory

(continued from page 24)

with the β/ϕ ratio is shown in Fig. 5. On applying the "Infinite Slope" analysis (Haefeli, 1948) to a dry cohesionless soil, Eqn. *12a*, it can be shown that for a factor of safety of not less than 1.5, β must be less than 2/3 ϕ.

$$F = \tan\phi/\tan\beta \qquad \ldots (12a)$$

$$\text{For } F > 1.5,\ \beta < 2/3\ \phi \qquad \ldots (12b)$$

It is worth mentioning that Rankine's expression for K_a for sloping fill retained by a smooth wall can lead to gross error (Müller-Breslau loc. cit). In his theory, Rankine (1857) deduced that the line of active thrust acts parallel to the surface of the fill; this implies that the thrust at the back of the wall can be resolved into two components, one normal to the back of the wall and one parallel to the back of the wall. Since the back of the wall is assumed smooth, obviously the shear stress parallel to the back of the wall cannot be mobilised. The error involved in the use of Rankine's theory is shown in Fig. 6, where the error is expressed as the difference of the Coulomb active pressure coefficient and the normal component of the Rankine coefficient, as a percentage of the Coulomb value.

Conclusions

Although two hundred years old, Coulomb's theory still renders acceptable values for active earth pressure, but the Coulomb expressions are cumbersome and tedious to evaluate. Contemporary researchers have shown that the inclinations of the rupture surfaces for rough walls and walls with sloping fills are, overall, different from those derived in the Coulomb-Mayniel and Coulomb-Müller-Breslau solutions. By substituting $45^\circ + \phi/2$ for the inclinations of the failure planes in these two solutions, a considerably simplified expression results. These simplified expressions may be used without significant loss of accuracy. It is shown that for the case of sloping fill behind a smooth wall, the use of the Rankine theory can lead to large error.

References

Broms, B. & Stille, H. (1976) "Failure of anchored sheet pile walls". Jl. Geot. Eng. Div. ASCE., Vol. **102**. GT3.

Caquot, A. & Kerisel, J. (1956); Traite de mecanique des sols. 3rd Ed. Gauthier-Villars Paris.

Coulomb, C. A. (1776): "Essai sur une application des regles de maximis et minimis a quelques problems de statique relatifs a l'architecture". Math. Mems.—Acad. Roy des Sciences Vol. **7**

Francois, (1820): "Recherches sur la poussee des terres" Memorial de l'officier du Genie. No. 4.

Haefeli, R. (1948): "The stability of slopes acted upon by parallel seepage". Proc. 2nd. Int. Conf. Soil Mech. and Fndn. Engng., Vol. **1**.

Kotter, F. (1903): "Die bestimmung des druckes an gekrummten gleitflächen" Sitzungsber Kgl. Preuss. Akad. der Wiss. Berlin.

Krey, H. (1936): "Erddruck, erdwiderstand und tragfähigkeit des baugrandes" Ernst. Berlin.

Mayniel, K. (1808): "Traite analytique et experimental de la poussee des terres". Paris.

Muller-Breslau (1906): "Erddruck auf stützmauern". Kroner, Stuttgart.

Prandtl, L. (1920): "Uber die härte plastischer Körper". Nachr. d. Ges. d. Wiss. Göttingen.

Prandtl, L. (1927): "Uber die eindringungsfestigkeit plastischer baustoffe und dis festigkeit des schneiden". Zeitschr. F. angew. Math. und Mech, Febr. 1927.

Prony, (1802): "Recherches sur la Poussee des Terres". Bulletin Paris Societie Philomatique Vol. **1**.

Rankine, W. J. M. (1857): "On the stability of loose earth". Trans. Roy. Soc. Lon. Vol. **147**.

Sokolovskii, V. V. (1942): "Statika Sypuchei sredy", Izd. Aka. Nauk. SSSR.

Woltmann, R. (1794): Hydraulische Architectur—Vol. **3**.